General Relativity $\boxed{\text{Week 1}}$

General Relativity: Theory of gravity, in which space & time are jointly modeled by M^{3+1}

Equiped with Lorentzian metric g : (M, g) Lorentzian manifold

$\rightsquigarrow$ Causal structure

"The geometry tells matter how to move, matter dictates how spacetime is curved" (Wheeler)

$$Ric - \frac{1}{2} R g = 8\pi T$$

Geometry ↗ ↖ Matter

Aim of this course: Understand this equation.

- Half of the course: Geometry / the other half: PDEs.

Riemannian metric: Modeled on Euclidean inner product space.

Lorentzian metric: " " Lorentzian inner product space

Definition: Let V be an $n+1$ dimensional vector space.

A Lorentzian inner product m on V is a map $m: V \times V \longrightarrow \mathbb{R}$

which is
1) bilinear
2) symmetric
3) non-degenerate, i.e. if $m(v, w) = 0 \ \forall w \in V \Rightarrow v = 0$
4) Has signature $(1, n)$ (or $-++\dots+$)
   i.e. the maximum dimension of subspace $W \subset V$ with $m|_W$ positive is n.

Example: On $\mathbb{R}^2$: $m_1(v,w) = - v_0 w_0 + v_1 w_1$

$m_2(v,w) = v_0 w_1 + v_1 w_0$

Given a basis $e_0, \dots, e_n$ of V: The matrix $m_{\alpha\beta} = m(e_\alpha, e_\beta)$ is symmetric and invertible

↖ equivalent to m being non-degenerate

and has 1 negative eigenvalue, n positive eigenvalues

If $X = X^\alpha e_\alpha$, $Y = Y^\beta e_\beta$ (Einstein summation notation)

then $m(X,Y) = m_{\alpha\beta} X^\alpha Y^\beta$

Proposition: There exists a basis $(e_0, \dots, e_n)$ in which

$$m_{00} = -1 \ , \ m_{ij} = \delta_{ij} \ , \ m_{0i} = 0$$

(Convention on indices: Greek letters $\alpha, \beta, \gamma, \dots = 0, 1, \dots, n$

Latin letters $i, j, k, \dots = 1, 2, \dots n$)

So every (V, m) is isometric to $(\mathbb{R}^{n+1}, \eta)$, $\eta = \begin{pmatrix} -1 & 0 & \cdots \\ & +1 & & 0 \\ 0 & & \ddots & \\ \vdots & 0 & & +1 \end{pmatrix}$

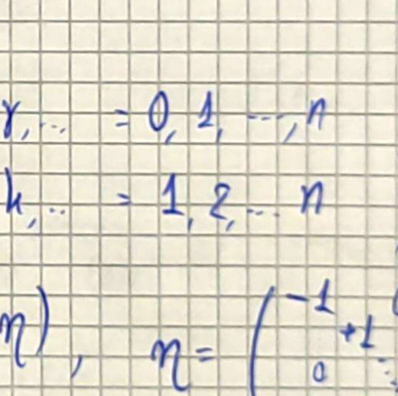
(Inner products of more general signature (p,q) are called pseudo-Euclidean)

Def: Let (V,m) be a Lorentzian inner product space.

A $v \in V / \{0\}$ is called:

- timelike if $m(v,v) < 0$
- null if $m(v,v) = 0$
- spacelike if $m(v,v) > 0$

$\Big\}$ causal



Convention: $v = 0$ is space like

Light cone: $N = \{ v \in V / \{0\} : m(v,v) = 0 \}$

Time cone: $I = \{ v \in V / \{0\} : m(v,v) < 0 \}$

Two components

In the exercises: We will show that if v is timelike an $w \perp v \Rightarrow$
$\Rightarrow$ w is spacelike

Based on this property, one can prove:

Prop: Gram-Schmidt process: Let v be a timelike vector. We can always find an orthonormal basis $\{e_0, e_1, \dots, e_n\}$ (i.e $m(e_\alpha, e_\beta) = \eta_{\alpha\beta}$) with $e_0 = \lambda \cdot v$ for some $\lambda > 0$.

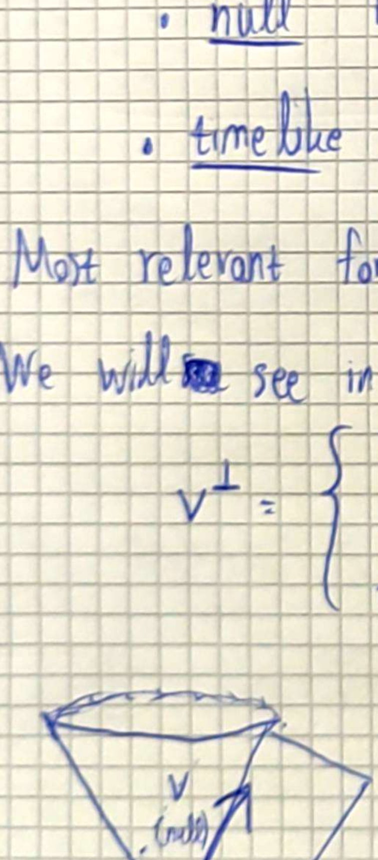
Def: A subspace $W \subseteq V$ is called:

- spacelike if $m|_W$ is positive definite
- null if $m|_W$ is degenerate
- timelike if $m|_W$ is Lorentzian

Most relevant for us: The cases $\dim W = 1$ or $\operatorname{codim} W = 1$

We will see in the exercises that if $v \in V \setminus \{0\}$

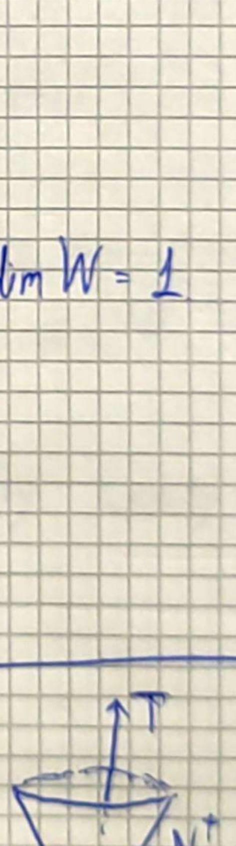
$$v^\perp = \begin{cases} \text{spacelike, if } v \text{ is timelike,} \\ \text{null, if } v \text{ is null, with } v \in v^\perp \\ \text{timelike, if } v \text{ is spacelike} \end{cases}$$



Null cone N has two components: Pick one, Call it future light cone N^+

Same as picking a timelike direction T:

$N^+ = \{ v \in N : m(v,T) < 0 \}$



Lemma: If v_1, v_2 are timelike vectors in the same timecone, then $m(v_1, v_2) < 0$.

Proof:

Let T be the chosen timelike vector defining the future cone.

Choose an orthonormal basis $e_0, \dots, e_n$ so that $e_0 = \lambda T$, $\lambda > 0$.

Then: v_1, v_2 in the same timecone as T

$$\Rightarrow m(v_1, T),\ m(v_2, T) < 0 \underset{\substack{m_{\alpha\beta} = \eta_{\alpha\beta} \\ \text{and } T = \frac{1}{\lambda} e_0}}{\Longrightarrow} v_1^0,\ v_2^0 > 0. \quad ①$$

v_1, v_2 timelike $\Rightarrow$

$$\Rightarrow m(v_1, v_1),\ m(v_2, v_2) < 0 \longrightarrow \begin{aligned} (v_1^0)^2 &> \sum_{i=1}^{n} (v_1^i)^2 \\ (v_2^0)^2 &> \sum_{i=1}^{n} (v_2^i)^2 \end{aligned} \quad ②$$

Then:

$$m(v_1, v_2) = -v_1^0 v_2^0 + \sum_{i=1}^{n} v_1^i v_2^i \overset{①,②}{<} -\sqrt{\sum_{i=1}^{n} (v_1^i)^2} \cdot \sqrt{\sum_{i=1}^{n} (v_2^i)^2} + \sum_{i=1}^{n} v_1^i v_2^i$$

$$\leq 0 \text{ by Cauchy-Schwarz}$$

So $m(v_1, v_2) < 0$ ☒

Def: $|v| = \sqrt{|m(v,v)|}$.

Proposition (proof in the exercises).

Let v, w be timelike vectors.

1) $|m(v,w)| \geq |v| \cdot |w|$, equality only if colinear (reverse Cauchy-Schwarz inequality)

2) If v, w in the same timecone: $|v + w| \geq |v| + |w|$, equality only if colinear

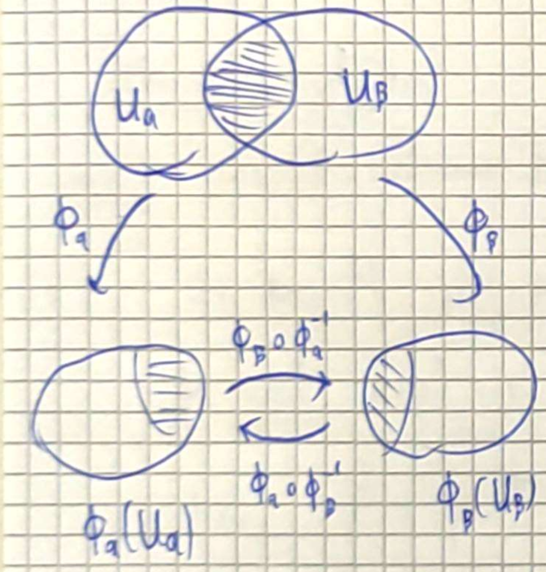
Def: If v, w timelike, in the same timecone.

Hyperbolic angle $q = \angle v,w$:

$$m(v,w) = -|v||w|\cosh(q)$$

Smooth manifolds

Def: A smooth manifold M of dimension m is a Hausdorff, second countable topological space, together with a collection of open sets $\{U_a\}_{a\in A}$ and homeomorphisms $\varphi_a : U_a \longrightarrow \varphi_a(U_a) \subseteq \mathbb{R}^n$



such that:

1) $M = \bigcup_{a\in A} U_a$

2) If $U_a \cap U_\beta \neq \emptyset$, then the map

$$\varphi_a\circ\varphi_\beta^{-1} : \underset{\subseteq\mathbb{R}^n}{\varphi_\beta(U_a\cap U_\beta)} \longrightarrow \underset{\subseteq\mathbb{R}^n}{\varphi_a(U_a\cap U_\beta)}$$

is a <u>diffeomorphism</u> of domains in $\mathbb{R}^n$

3) $\{(U_a, \varphi_a)\}_{a\in A}$: <u>maximal</u> collection with these properties

~~(differentiable structure)~~

- (U_a, φ_a): Coordinate chart

  φ_a^{-1} : parametrization

local coordinates: $(x^1, \ldots, x^m)$

- A collection $\{(U_a, \varphi_a)\}_a$ satisfying (1) & (2): <u>smooth atlas</u>
- A maximal smooth atlas: Differentiable structure.

From the above assumptions: If $\{V_a\}_a$ is a collection of open sets covering M, then there exists a *partition of unity* $\{\chi_\beta\}_\beta$ subordinate to $\{V_a\}_a$, i.e:

- $\forall \beta$, $\chi_\beta : M \longrightarrow [0,1]$ is a smooth function
- $\forall \beta$, $\exists\, a = a(\beta)$ such that $\operatorname{supp} \chi_\beta \subseteq V_a$ *and*

  $\operatorname{supp} \chi_\beta$ is contained in the domain of a coordinate chart.
- $\forall \beta$, $\operatorname{supp} \chi_\beta \cap \operatorname{supp} \chi_\gamma \neq \emptyset$ for at most finitely many γ.
- $\sum_{\beta \in B} \chi_\beta = 1$.

Smooth functions:

Def: Let $F: M \longrightarrow N$ be a function. It is *smooth* (or C^k, etc) at $p \in M$ if its expression in local coordinates around p and $F(p)$ is smooth (respectively, C^k) at p.

i.e. $\tilde{F} = \psi \circ F \circ \phi^{-1}$ is smooth for any chart (U, ϕ) on M around p and (V, ψ) on N around $F(p)$

Note: Independent of the choice of coordinate charts.

Tangent space - Cotangent space:

Let M be a smooth manifold, $p \in M$.

Def: Tangent vector X_p at p: A linear map $X_p : C^\infty(M) \longrightarrow \mathbb{R}$

with the property (Leibniz rule):

$$X_p(f \cdot g) = X_p f \cdot g(p) + f(p) \cdot X_p g.$$

Tangent space T_pM: Set of tangent vectors at p.

- It's a vector space, $\dim T_pM = \dim M$.

If $(x^1, \ldots, x^m)$ local coordinates: $\frac{\partial}{\partial x^i}\Big|_p$ coordinate tangent vectors.

- Form a basis of T_pM
- Any tangent vector: $X = \overset{\swarrow \text{ indices up!}}{X^i} \frac{\partial}{\partial x^i}$

(Smooth) vector field: a choice of a tangent vector X at every $p \in M$, such that the components X^i with respect to any local coordinate chart are smooth functions.

Example: If $\gamma : (-\varepsilon, \varepsilon) \to M$ is a smooth curve,

$$\dot{\gamma}(0) \in T_{\gamma(0)}M \quad , \quad \dot{\gamma}(0)(f) = \frac{d}{dt}(f \circ \gamma(t))\Big|_{t=0}$$

In local coordinates: $\dot{\gamma}^i = \frac{d\gamma^i}{dt}$

Cotangent space:

T_p^*M: linear functionals $\omega : T_pM \to \mathbb{R}$

If $f \in C^\infty(M)$: $df_p(v_p) = v_p(f)$

Basis: $\{dx^i\}_{i=1}^m \longleftarrow$ Dual basis of $\left\{\frac{\partial}{\partial x^j}\right\}_{j=1}^m$

i.e. $dx^i\left(\frac{\partial}{\partial x^j}\right) = \delta^i_j$

Change of coordinates formula:

$$(x^1, \dots, x^m) \longrightarrow (y^1, \dots, y^m)$$

$$\boxed{\begin{aligned} dy^i &= \frac{\partial y^i}{\partial x^j}\, dx^j \\ \frac{\partial}{\partial y^i} &= \frac{\partial x^j}{\partial y^i} \cdot \cancel{d}\, \frac{\partial}{\partial x^j} \end{aligned}}$$

Lorentzian manifolds:

Def: Let M be a smooth manifold. A Lorentzian metric on M is an assignment g_p of a Lorentzian inner product on T_pM for all $p \in M$, such that $g_{\alpha\beta} = g\left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}\right)$ are smooth functions in any local coordinate chart.

- g: Symmetric $(0,2)$-tensor field
- $g(X,Y) = g_{\alpha\beta} X^\alpha Y^\beta$

We will write $g = g_{\alpha\beta}\, dx^\alpha dx^\beta = g_{\alpha\beta}\, dx^\alpha \otimes dx^\beta$

- A vector field $X \in \Gamma(M)$ is timelike / null / spacelike if, $\forall p \in M$, X_p is " / " / "
- A submanifold $S \subseteq M$ is timelike / null / spacelike if, $\forall p \in S$, T_pS is a " / " / " subspace of T_pM.
- A C^1 curve γ is timelike / null / spacelike if $\dot{\gamma}$ is " / " / ".